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Analytical investigation of Couette flow in a composite channel partially filled with a porous medium and partially with a clear fluid

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1. INTRODUCTION

Convective heat transfer in porous media is a subject of growing interest. This is because of many important engineering applications of porous media. Such applications can be found in heat exchanges, energy storage units, chemical reactors, heat pipes, electronic cooling and ceramic processing. One of important and fundamental fluid flow situations in porous media is Couette flow which can occur, for example, between two parallel flat plates, one of which is at rest, and the other is moving in its own plane with a constant velocity, $\tilde{u}_w$. Since in Couette flow velocity of the moving plate can be large, and also viscous forces in the boundary layer near the moving plate can be significant, to obtain a correct description of the flow it can be important to account for non-Darcian effects, namely, for the inertial (Forchheimer) and for the viscous (Brinkman) effects [1].

Investigations of heat transfer in Couette flow through a porous medium are limited to the case of a Brinkman–Darcy porous medium [2, 3]. In an insightful investigation by Nakayama [4], analytical solutions for different situations of Couette flow through an inelastic porous medium, including a porous medium described by the Brinkman–Forchheimer extension of the Darcy law, are obtained. However, results of ref. [4] are limited to the fluid flow analysis only, and no investigation of heat transfer is made in this reference. To the best of the author's knowledge, no attempt to analyze heat transfer in Couette flow through a Brinkman–Forchheimer–Darcy porous medium has yet been made.

In [4] it is assumed that the channel is completely filled with a porous medium which is at rest, and there is no gap between the porous medium and the moving plate. Such a geometry can result in large friction forces between the porous matrix and the moving plate. This can lead to a damage of the porous matrix. In practical situations, it is necessary to assume that between a porous medium and a moving plate there is a gap filled with a clear fluid. Even if this gap is small, its influence on heat transfer can be significant. Accounting for this gap essentially complicates the problem, because it is necessary to analyze fluid flow and heat transfer in a composite channel, which is partially filled with a fluid saturated porous medium, and partially with a clear fluid. In solving this problem, correct boundary conditions at the porous medium/clear interface are important. In this research we utilize the boundary conditions at the interface suggested in Ochoa-Tapia and Whitaker [5, 6].

2. ANALYSIS

Figure 1 depicts the schematic diagram of the problem. Steady flow in a composite channel bounded by two infinite parallel plates is considered. The distance between the plates is H. The lower part of the channel is occupied by a clear fluid while the upper part is occupied by a fully saturated porous medium with uniform permeability. The upper plate and the porous medium are fixed, while the lower plate moves with a constant velocity, $\tilde{u}_w$. The fluid flow thus occurs due to a moving plain plate which is separated from the porous medium by a gap filled with a clear fluid of the thickness δ. It is assumed that a uniform heat flux is imposed at the moving plate while the fixed plate is insulated. The governing equations for this problem can be presented as:

$$\mu_f \frac{d^2\tilde{u}_f}{d\tilde{y}^2} = 0 \quad -\delta \leqslant \tilde{y} \leqslant 0 \tag{1}$$

$$\mu_{eff}\frac{d^2\tilde{u}_f}{d\tilde{y}^2} - \frac{\mu_f}{K}\tilde{u}_f - \frac{\rho_f F}{K^{1/2}}\tilde{u}_f^2 = 0 \quad 0 \leqslant \tilde{y} \leqslant L \tag{2}$$

$$\rho_f c_f \tilde{u}_f \frac{\partial \tilde{T}}{\partial \tilde{x}} = k_f \frac{\partial^2 \tilde{T}}{\partial \tilde{y}^2} \quad -\delta \leqslant \tilde{y} \leqslant 0 \tag{3}$$

$$\rho_f c_f \tilde{u}_f \frac{\partial \tilde{T}}{\partial \tilde{x}} = k_{eff} \frac{\partial^2 \tilde{T}}{\partial \tilde{y}^2} \quad 0 \leqslant \tilde{y} \leqslant L. \tag{4}$$

Equation (1) is a momentum equation for the clear fluid region while equation (2) is a momentum equation for the porous region (the Brinkman–Forchheimer-extended Darcy equation). Equations (3) and (4) are the energy equations for the clear fluid and porous regions, respectively. Following [7–9], in equations (3) and (4) it is assumed that the longitudinal heat conduction term is negligible. Also, in the porous region the local thermal equilibrium assumption between the fluid and solid phases is invoked.

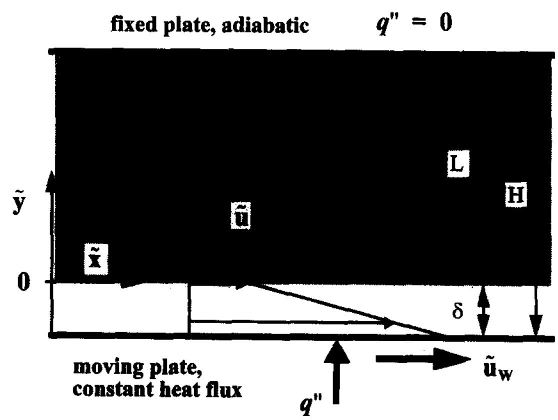

Fig. 1. Schematic diagram of the problem.

NOMENCLATURE

A	parameter defined in equation (23), $Re_H F/Da_H^{1/2}$
B	parameter defined in equation (23), $1/Da_H$
c_f	specific heat of the fluid [J/(kg K)]
D	parameter defined in equation (23), $[1+(1+(2/3)(A/B)u_i)^{1/2}]/[1-(1+(2/3)(A/B)u_i)^{1/2}]$
Da_H	Darcy number $[K/H^2]$
F	Forchheimer coefficient
H	thickness of the channel, $L+\delta$ [m]
k_f	thermal conductivity of the fluid [W/(m K)]
k_{eff}	effective thermal conductivity of the porous medium [W/(m K)]
K	permeability of the porous medium [m^2]
L	thickness of the porous layer [m]
Nu	Nusselt number at the isoflux plate, $Hq''/[k_f(\tilde{T}_w-\tilde{T}_m)]$
R	thermal conductivity ratio, k_{eff}/k_f
Re_H	Reynolds number, $\rho_f\tilde{u}_w H/\mu_f$
T	dimensionless temperature, $[\tilde{T}-\tilde{T}_w]/[\tilde{T}_m-\tilde{T}_w]$
T_i	dimensionless temperature at the clear fluid/porous medium interface
$\tilde{T}$	intrinsic average temperature [K]
$\tilde{T}_m$	mean temperature, $(1/HU)\int_{-\delta}^{L}\tilde{u}_f\tilde{T}\,d\tilde{y}$ [K]
$\tilde{T}_w$	temperature at the isoflux plate [K]
u	dimensionless velocity, $\tilde{u}_f/\tilde{u}_w$
u_i	dimensionless velocity at the clear fluid/porous medium interface
$\tilde{u}_f$	filtration (seepage) velocity [m s^{-1}]
$\tilde{u}_w$	velocity of the lower plate [m s^{-1}]
$\tilde{U}$	mean flow velocity, $(1/H)\int_{-\delta}^{L}\tilde{u}_f\,d\tilde{y}$ [m s^{-1}]
$\tilde{x}$	streamwise coordinate [m]
$\tilde{y}$	transverse coordinate [m]
y	dimensionless transverse coordinate, $\tilde{y}/H$.

Greek symbols

β	the adjustable coefficient in the stress jump boundary condition
γ	constant, $(\mu_{eff}/\mu_f)^{1/2}$
δ	thickness of the fluid layer, m
ε	porosity
μ_f	fluid viscosity [kg m^{-1} s^{-1}]
μ_{eff}	effective viscosity in the Brinkman term for the porous region [kg m^{-1} s^{-1}]
ρ_f	density of the fluid [kg m^{-3}].

Equations (1)–(4) are subject to the following boundary conditions:

$$\tilde{u}_f=\tilde{u}_w \quad k_f\frac{\partial\tilde{T}}{\partial\tilde{y}}=q'' \quad \text{at } \tilde{y}=-\delta \tag{5}$$

$$\tilde{u}_f=0 \quad \frac{\partial\tilde{T}}{\partial\tilde{y}}=0 \quad \text{at } \tilde{y}=L \tag{6}$$

$$\begin{gathered}\tilde{u}_f|_{\tilde{y}=-0}=\tilde{u}_f|_{\tilde{y}=+0}\\ \mu_{eff}\frac{d\tilde{u}_f}{d\tilde{y}}\bigg|_{\tilde{y}=+0}-\mu_f\frac{d\tilde{u}_f}{d\tilde{y}}\bigg|_{\tilde{y}=-0}=\beta\frac{\mu_f}{K^{1/2}}\tilde{u}_f|_{\tilde{y}=0}\\ k_{eff}\frac{d\tilde{T}}{d\tilde{y}}\bigg|_{\tilde{y}=+0}=k_f\frac{d\tilde{T}}{d\tilde{y}}\bigg|_{\tilde{y}=-0} \quad \text{at } \tilde{y}=0.\end{gathered} \tag{7}$$

The first two equations in (7) present continuity of the seepage velocity and the stress jump condition at the interface suggested in refs. [5, 6]. In refs. [5, 6] the inertial forces were not included in the analysis. At the same time, the stress jump boundary condition contains an empirical constant, β, which is to be determined experimentally. This permits necessary flexibility in modeling the interface and in adjusting these conditions to experimental data. Therefore, we believe that these conditions are appropriate to match the Stokes and the Brinkman–Forchheimer equations at the porous medium/clear fluid interface [10].

Introducing dimensionless variables and utilizing the isoflux condition, the momentum and energy equations, equations (1)–(4), take the following form:

$$\frac{d^2u}{dy^2}=0 \quad -\frac{\delta}{H}\leqslant y\leqslant 0 \tag{8}$$

$$\gamma^2\frac{d^2u}{dy^2}-\frac{1}{Da_H}u-\frac{Re_H F}{Da_H^{1/2}}u^2=0 \quad 0\leqslant y\leqslant\frac{L}{H} \tag{9}$$

$$\frac{d^2T}{dy^2}=-Nu\frac{\tilde{u}_w}{\tilde{U}}u \quad -\frac{\delta}{H}\leqslant y\leqslant 0 \tag{10}$$

$$R\frac{d^2T}{dy^2}=-Nu\frac{\tilde{u}_w}{\tilde{U}}u \quad 0\leqslant y\leqslant\frac{L}{H} \tag{11}$$

where

$$y=\frac{\tilde{y}}{H},\quad u=\frac{\tilde{u}_f}{\tilde{u}_w},\quad T=\frac{\tilde{T}-\tilde{T}_w}{\tilde{T}_m-\tilde{T}_w},\quad R=\frac{k_{eff}}{k_f},\quad \gamma=\left(\frac{\mu_{eff}}{\mu_f}\right)^{1/2} \tag{12}$$

$$Re_H=\frac{\rho_f\tilde{u}_w H}{\mu_f},\quad Da_H=\frac{K}{H^2},\quad Nu=\frac{Hq''}{k_f(\tilde{T}_w-\tilde{T}_m)}. \tag{13}$$

The mean flow velocity, $\tilde{U}$, in equations (10) and (11) is determined by the following equation:

$$\tilde{U}=\frac{1}{H}\int_{-\delta}^{L}\tilde{u}_f\,d\tilde{y} \tag{14}$$

and the mean temperature, $\tilde{T}_m$, in equations (12) and (13) is determined by the following equation:

$$\tilde{T}_m=\frac{1}{H\tilde{U}}\int_{-\delta}^{L}\tilde{u}_f\tilde{T}\,d\tilde{y}. \tag{15}$$

After the velocity distribution is found from equations (8) and (9) and the temperature distribution is found from equations (10) and (11) as a function of the Nusselt number, the value of the Nusselt number can be found by substituting u and T in the following compatibility condition:

$$\frac{\tilde{u}_w}{\tilde{U}}\int_{-\delta/H}^{L/H}Tu\,dy=1. \tag{16}$$

Equations (8)–(11) must be solved subject to the following boundary conditions:

$$u=1 \quad T=0 \quad \text{at } y=-\frac{\delta}{H} \tag{17}$$

$$u = 0 \quad \frac{\partial T}{\partial y} = 0 \quad \text{at } y = \frac{L}{H} \tag{18}$$

$$u|_{y=+0} = u|_{y=-0}$$

$$\gamma^2 \left.\frac{du}{dy}\right|_{y=+0} - \left.\frac{du}{dy}\right|_{y=-0} = \beta Da_H^{-1/2} u|_{y=0}$$

$$R \left.\frac{dT}{dy}\right|_{y=+0} = \left.\frac{dT}{dy}\right|_{y=-0} \quad \text{at } y = 0. \tag{19}$$

Exact solution of equations (8)–(11) with boundary conditions (17)–(19) is not available in a closed form. However, for most practical applications it can be assumed that the momentum boundary layer which forms near the moving plate does not reach the fixed plate. This allows to replace the no-slip boundary condition at the fixed plate given by the first of equations (18) with the following boundary condition outside the momentum boundary layer:

$$u \to 0 \quad \text{and} \quad \frac{\partial u}{\partial y} \to 0 \quad \text{as } y \to \infty. \tag{20}$$

A similar assumption was utilized by Vafai and Kim [11] in obtaining their analytical solution for the forced convection in a parallel-plate channel filled with a porous medium. They assumed that the momentum boundary layer does not reach the center of the channel. A comparison of the solution obtained by Vafai and Kim [11] against a full numerical solution is presented in [12]. This comparison has shown that this is a good assumption as long as $Da_H < 1$, which is the case for most practical porous media.

If the momentum boundary layer does not reach the fixed plate, equations (8) and (9) can be integrated analytically and the solution for the fluid flow velocity takes the following form:

$$u = u_i - (1-u_i)\frac{H}{\delta} y \qquad -\frac{\delta}{H} \leqslant y \leqslant 0 \tag{21}$$

$$u = \frac{3}{2}\frac{B}{A}\left\{\left[\frac{D\exp\left(\frac{B^{1/2}}{\gamma}y\right)-1}{D\exp\left(\frac{B^{1/2}}{\gamma}y\right)+1}\right]^2 - 1\right\} \qquad 0 \leqslant y \leqslant \frac{L}{H} \tag{22}$$

where

$$A = \frac{Re_H F}{Da_H^{1/2}}, \quad B = \frac{1}{Da_H}, \quad D = \frac{1+\left(1+\frac{2}{3}\frac{A}{B}u_i\right)^{1/2}}{1-\left(1+\frac{2}{3}\frac{A}{B}u_i\right)^{1/2}} \tag{23}$$

and the dimensionless velocity at the clear fluid/porous medium interface, u_i, can be found from the following transcendental equation

$$-\gamma u_i\left[\frac{2}{3}Au_i + B\right]^{1/2} + (1-u_i)\frac{H}{\delta} = \beta B^{1/2} u_i. \tag{24}$$

The mean flow velocity, $\bar{U}$, can then be calculated from the following equation

$$\frac{\bar{U}}{\tilde{u}_w} = u_i\frac{\delta}{H} + \frac{1}{2}(1-u_i)\frac{\delta}{H} + 6\gamma\frac{B^{1/2}}{A}\left[\frac{1}{1+D\exp\left[\frac{B^{1/2}}{\gamma}\frac{L}{H}\right]} - \frac{1}{1+D}\right]. \tag{25}$$

With the flow velocity given by equations (21) and (22), equations (10) and (11) can also be integrated analytically. This results in the following temperature distribution:

$$T = Nu\frac{\tilde{u}_w}{\bar{U}}(1-u_i)\frac{H}{\delta}\frac{y^3}{6} - Nu\frac{\tilde{u}_w}{\bar{U}}u_i\frac{y^2}{2} - \left[\frac{1}{6}Nu\frac{\tilde{u}_w}{\bar{U}}(1-u_i)\frac{\delta}{H} + \frac{1}{2}Nu\frac{\tilde{u}_w}{\bar{U}}u_i\frac{\delta}{H} - T_i\frac{H}{\delta}\right]y + T_i \qquad -\frac{\delta}{H} \leqslant y \leqslant 0 \tag{26}$$

$$T = \frac{6\gamma}{A}\frac{Nu}{R}\frac{\tilde{u}_w}{\bar{U}}\left\{\gamma \ln\left[\frac{1+D\exp\left[\frac{B^{1/2}}{\gamma}y\right]}{1+D}\right] - \frac{yB^{1/2}D\exp\left[\frac{B^{1/2}}{\gamma}\frac{L}{H}\right]}{1+D\exp\left[\frac{B^{1/2}}{\gamma}\frac{L}{H}\right]}\right\} + T_i \qquad 0 \leqslant y \leqslant \frac{L}{H} \tag{27}$$

where the dimensionless temperature at the clear fluid/porous medium interface, T_i, can be found as

$$T_i = Nu\frac{\tilde{u}_w}{\bar{U}}G \tag{28}$$

where

$$G = 6\frac{B^{1/2}}{A}\gamma\frac{\delta}{H}\left[\frac{1}{1+D\exp\left[\frac{B^{1/2}}{\gamma}\frac{L}{H}\right]} - \frac{1}{1+D}\right] + \frac{1}{6}(1-u_i)\left(\frac{\delta}{H}\right)^2 + \frac{1}{2}u_i\left(\frac{\delta}{H}\right)^2 \tag{29}$$

Finally, if the velocity and temperature distributions are given by equations (21)–(22) and (26)–(27), respectively, equation (16) can also be solved analytically. The solution for the Nusselt number is then obtained as:

$$Nu = \left(\frac{\bar{U}}{\tilde{u}_w}\right)^2 \Xi^{-1} \tag{30}$$

where

$$\Xi = -\frac{1}{6}u_i\left(\frac{\delta}{H}\right)^3 + \frac{u_i}{2}\left(\frac{\delta}{H}\right)^2\left[\frac{1}{6}(1-u_i)\frac{\delta}{H} + \frac{1}{2}u_i\frac{\delta}{H} - G\frac{H}{\delta}\right] + u_i G\frac{\delta}{H}$$
$$-\frac{1}{30}(1-u_i)^2\left(\frac{\delta}{H}\right)^3 + \frac{(1-u_i)}{3}\left(\frac{\delta}{H}\right)^2 \times\left[\frac{1}{6}(1-u_i)\frac{\delta}{H} + \frac{1}{2}u_i\frac{\delta}{H} - G\frac{H}{\delta}\right] + \frac{1}{2}\frac{\delta}{H}G(1-u_i)$$

$$+36\frac{B}{A^2}\frac{\gamma^2}{R}\frac{L}{H}\frac{D\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}{1+D\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}$$

$$\times\left\{1-\frac{1}{1+\mathrm{D}\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}\right\}$$

$$-36\frac{B^{1/2}}{A^2}\frac{\gamma^3}{R}\left[\frac{1}{1+D}-\frac{1}{1+D\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}\right]$$

$$-6\gamma\frac{B^{1/2}}{A}G\left[\frac{1}{1+D}-\frac{1}{1+D\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}\right]$$

$$+36\frac{B^{1/2}}{A^2}\frac{\gamma^3}{R}\frac{1-D\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}{1+D\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}$$

$$\times\ln\left[\frac{1+D\exp\left[\dfrac{B^{1/2}}{\gamma}\dfrac{L}{H}\right]}{1+D}\right]. \tag{31}$$

Figure 2 (top) shows the velocity and temperature distributions in the channel. This figure is computed utilizing equations (21)–(22) and (26)–(27) for the following parameter values: $A = 10^2$, $\gamma = 1$, $\beta = 0$, $R = 1$ and $\delta/H = 0.1$. The lower plate is a moving plate, and the dimensionless fluid velocity at $y = -\delta$ equals unity. Because the porous medium creates resistance to the fluid flow, the fluid velocity quickly decreases with distance from the moving plate. A

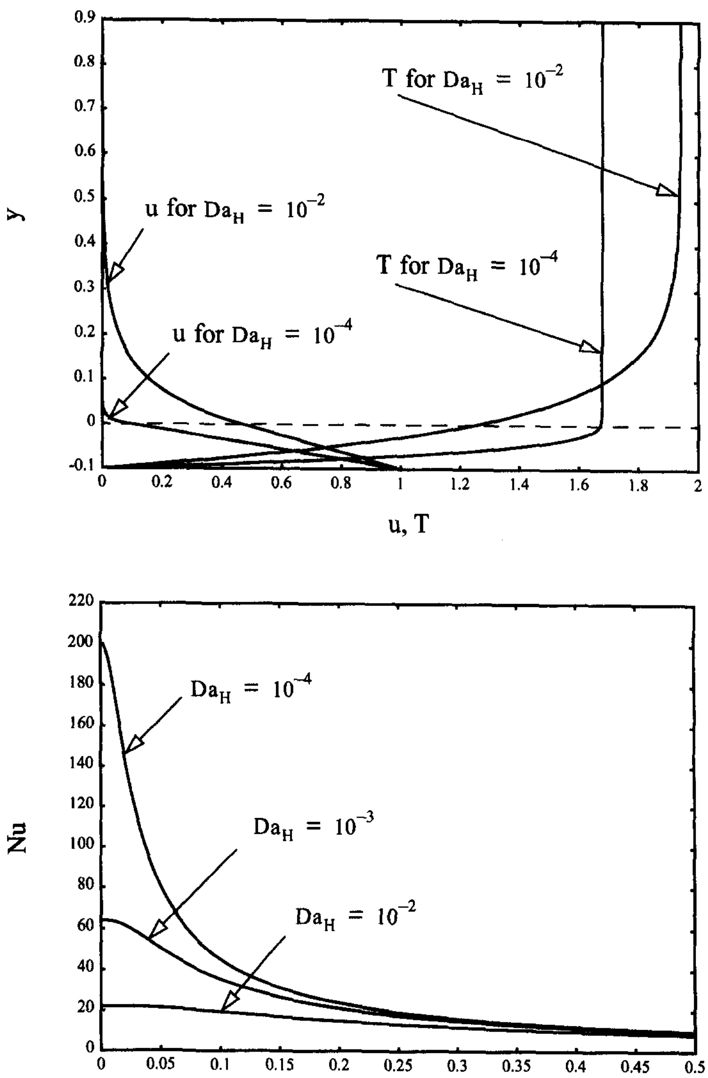

Fig. 2. Velocity and temperature distributions (top) and Nusselt number as a function of the gap size between the porous medium and the moving plate (bottom).

decrease in the Darcy number translates into a decrease in permeability of the porous medium and therefore leads to a faster velocity decrease with an increase in the coordinate y. It can be seen that for both $Da_H = 10^{-4}$ and $Da_H = 10^{-2}$ the momentum boundary layer does not reach the fixed plate, therefore no-slip boundary condition at the upper (fixed) plate is perfectly satisfied.

According to Fig. 2 (top), the dimensionless temperature in the porous region of the channel for a small value of the Darcy number ($Da_H = 10^{-4}$) is almost constant. This is because for $Da_H = 10^{-4}$ there is almost no fluid flow in the porous region, and heat transfer in the porous region is caused almost only by thermal conductivity. Since the upper plate is insulated, the temperature is almost constant. It can also be seen that the temperature gradient at the moving plate increases with a decrease in the Darcy number.

Figure 2 (bottom) shows the dependence of the Nusselt number on the size of the gap between the porous medium and the moving plate. This figure is computed utilizing equations (30) and (31) for the following parameter values: $A = 10^2$, $\gamma = 1$, $\beta = 0$ and $R = 1$. It can be seen that the Nusselt number increases with a decrease in the width of the gap and with a decrease in the Darcy number. This is because the decrease in the width of the gap as well as the decrease in the permeability of the porous medium cause the temperature gradient near the moving plate to increase. This results in a larger value of the Nusselt number.

3. CONCLUSIONS

A problem of fluid flow and heat transfer in Couette flow through a composite channel, which is partially filled with a fluid saturated porous medium, and partially with a clear fluid, is considered. The flow in the porous region is described by the Brinkman–Forchheimer-extended Darcy equation. For the analysis of heat transfer insulated fixed plate and isoflux moving plate are considered. This problem is solved under the boundary layer approximation. Analytical solutions for the flow velocity, temperature distribution and for the Nusselt number are obtained.

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REFERENCES

1. Nield, D. A. and Bejan, A., *Convection in Porous Media.* Springer, New York, 1992.
2. Bhargava, S. K. and Sacheti, N. C., Heat transfer in generalized Couette flow of two immiscible Newtonian fluids through a porous channel: use of Brinkman model. *Indian Journal of Technology*, 1989, **27,** 211–214.
3. Daskalakis, J., Couette flow through a porous medium of a high Prandtl number fluid with temperature-dependent viscosity. *International Journal of Energy Research*, 1990, **14,** 21–26.
4. Nakayama, A., Non-Darcy Couette flow in a porous medium filled with an inelastic non-Newtonian fluid. *ASME Journal of Fluids Engineering*, 1992, **114,** 642–647.
5. Ochoa-Tapia, J. A. and Whitaker, S., Momentum transfer at the boundary between a porous medium and a homogeneous fluid—I. Theoretical development. *International Journal of Heat and Mass Transfer*, 1995, **38,** 2635–2646.
6. Ochoa-Tapia, J. A. and Whitaker, S., Momentum transfer at the boundary between a porous medium and a homogeneous fluid—II. Comparison with experiment. *International Journal of Heat and Mass Transfer*, 1995, **38,** 2647–2655.
7. Kaviany, M., Laminar flow through a porous channel bounded by isothermal parallel plates. *International Journal of Heat and Mass Transfer*, 1985, **28,** 851–858.
8. Nakayama, A., Koyama, H. and Kuwahara, F., An analysis on forced convection in a channel filled with a Brinkman–Darcy porous medium: exact and approximate solutions. *Wärme- und Stoffübertragung*, 1988, **23,** 291–295.
9. Cheng, P., Hsu, C. T. and Chowdhury, A., Forced convection in the entrance region of a packed channel with asymmetric heating. *ASME Journal of Heat Transfer*, 1988, **110,** 946–954.
10. Kuznetsov, A. V., Influence of the stress jump boundary condition at the porous-medium/clear-fluid interface on a flow at a porous wall. *International Communications in Heat and Mass Transfer*, 1997, **24,** 401–410.
11. Vafai, K. and Kim, S. J., Forced convection in a channel filled with a porous medium: an exact solution. *ASME Journal of Heat Transfer*, 1989, **111,** 1103–1106.
12. Vafai, K. and Kim, S. J., Discussion of the paper by A. Hadim, "Forced convection in a porous channel with localized heat sources". *ASME Journal of Heat Transfer*, 1995, **117,** 1097–1098.